

LEBESGUE-STIELTJES INTEGRAL EQUATIONS OF THE FIRST KIND

BY

CORINNE ROSE HATTAN

B.S., Ottawa University, 1925

M.A., University of Kansas, 1928

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN
MATHEMATICS IN THE GRADUATE SCHOOL
OF THE UNIVERSITY OF ILLINOIS, 1945

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I. INTRODUCTION

This paper is concerned with the solution of the Lebesgue-Stieltjes integral equation of the form

$$(1.1) \quad \int_{(D_x)} \left[\int_{(e_y)} L(x, y) du(e_y) \right] d\psi(e_x) = F(e_y)$$

where (D_x) , (D_y) , ... are closed sets of points which have interior points and frontiers of measure zero. (e_x) , (e_y) , ... are sets of points contained in (D_x) , (D_y) , ... respectively. These sets are Borel measurable, and differ only in notation. Points x , y , ... shall lie in sets (e_x) , (e_y) , ... respectively.

$u(e_x)$ shall be a completely additive, continuous, non-negative measure-function, which exists on Borel sets $(e_x) \subset (D_x)$. $F(e_y)$ shall be a continuous, additive function, and there shall exist a function $f(y) = dF(e_y)$, such that $f^2(y)$ is integrable (B, u) .

Further, for every finite n , it is assumed that there exists a continuous function $f_n(y)$ defined by

$$(1.2) \quad f_n(y) = \begin{cases} f(y) & \text{if } |f(y)| \leq n \\ n & \text{if } f(y) > n \\ -n & \text{if } f(y) < -n \end{cases}$$

One then defines $F_n(e_y) = \int_{(e_y)} f_n(y) du(e_y)$. For such functions

$\text{Lim } f_n(y) = f(y)$ and $\text{Lim } F_n(e_y) = F(e_y)$. One also defines

$$(1.3) \quad K(x, e_y) = \int_{(e_y)} L(x, y) du(e_y)$$

and equation (1.1) becomes

$$(1.4) \quad \int_{(D_x)} K(x, e_y) d\psi(e_x) = F(e_y).$$

Integrable symmetric kernels $L(x, y)$ shall be considered, but unless it is explicitly stated, it is not assumed that $L^2(x, y)$ is integrable (B, u) .

Functions $L_n(x, y)$ and $K_n(x, e_y)$ may be defined in a way analogous to the definition of $f_n(y)$ in (1.2) and $K(x, e_y)$ in (1.3). For such functions* $\text{Lim}_{n \rightarrow \infty} K_n(x, e_y) = K(x, e_y)$. Approximation equations for (1.1) or (1.4) are then

$$(1.5) \quad \int_{(D_x)} K_n(x, e_y) d\psi_n(e_x) = F_n(e_y).$$

* E. C. Titchmarsh, "Theory of Functions," Chap. 10.

All integrals are to be Lebesgue-Stieltjes integrals.* The solutions $\psi_n(e_x)$ of equations (1.5) and the solution $\psi(e_x)$ of (1.4) shall be set functions, which are continuous in Borel sets (e_x) . $d\psi(e_x)$ indicates the general derivative† of $\psi(e_x)$ with respect to the measure function $u(e_x)$. Where several variables occur in the function a subscript will be used.

II. PROPERTIES OF THE SPECTRAL FUNCTIONS

Associated with the kernel $K_n(x, e_y)$ of equations (1.5) there are for every n , sequences of real characteristic numbers and of characteristic functions: $\{\lambda_{n1}, \lambda_{n2}, \dots\}$, $\{\phi_{n1}(x), \phi_{n2}(x), \dots\}$ and $\{\psi_{n1}(e_x), \psi_{n2}(e_x), \dots\}$ respectively.* $\phi_{nk}(x)$ are orthonormal for every n , and $d\psi_{nk}(e_x) = \phi_{nk}(x)$.

The spectral function $\theta_n(x, y|\lambda)$, with parameter λ , is defined by

$$\theta_n(x, y|\lambda) = \begin{cases} \sum_{0 < \lambda_{nk} < \lambda} \phi_{nk}(x) \phi_{nk}(y) & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda = 0 \\ -\sum_{\lambda_{nk} \leq \lambda < 0} \phi_{nk}(x) \phi_{nk}(y) & \text{if } \lambda < 0. \end{cases}$$

Here the summation is on k for values of k such that λ_{nk} lies in the prescribed range. Similar definitions may be given for $\theta_n(x, e_y|\lambda)$ and $\theta_n(e_x, e_y|\lambda)$ and these are made so that

$$(2.1) \quad \theta_n(e_x, e_y|\lambda) = \int_{(e_x)} \theta_n(x, e_y|\lambda) du(e_x) = \int_{(e_y)} \int_{(e_x)} \theta_n(x, y|\lambda) du(e_x) du(e_y).$$

There exists‡ a subsequence $\{n_i\} \subset \{n\}$ and a function $\theta(e_x, e_y|\lambda)$ such that for sets (e_x) , (e_y) on whose frontiers the measure function vanishes, one has

$$(2.2) \quad \lim_{n_i \rightarrow \infty} \theta_{n_i}(e_x, e_y|\lambda) = \theta(e_x, e_y|\lambda).$$

The function $\theta(e_x, e_y|\lambda)$ is bounded and of bounded variation,‡ and a generalized Bessel's inequality

$$\int_{-\infty}^{\infty} d_{\lambda} \int_{(D_x)} h(x) d_x \left[\int_{(D_y)} \theta(e_x, e_y|\lambda) h(y) du(e_y) \right] du(e_x) \leq \int_{(D_x)} h^2(x) du(e_x)$$

will hold. When this relation holds with the equality sign, then almost everywhere, for all $f(x)$ for which $f^2(x)$ is integrable (B, u)

$$(2.3) \quad f(x) = d_x \int_{-\infty}^{\infty} d_{\lambda} \int_{(D_y)} d_y \theta(e_x, e_y|\lambda) f(y) du(e_y).$$

* W. J. Trjitzinsky, "Singular Lebesgue-Stieltjes Integral Equations," *Acta Mathematica*, 74:3-4(1942):197-310. Hereafter "(W.J.T.)" will be used when reference is made to this paper.

† S. Saks, "Theory of the Integral," Chap. 4.

‡ (W.J.T.) Sections 12 and 13.

III. RELATION BETWEEN EQUATIONS (1.1) AND (3.1)

W. J. Trjitzinsky* has discussed solutions of the equation

$$(3.1) \quad \int_{(D_x)} L(x, y) \phi(x) du(e_x) = f(y)$$

where the terms are defined as they are in this paper.

THEOREM. A solution of equation (1.1) exists and is given by

$$(3.2) \quad \psi(e_x) = \int_{(e_x)} \phi(x) du(e_x) + \Omega(e_x)$$

where $\Omega(e_x)$ is any singular function of (e_x) provided

$$(\alpha) \quad L^2(y) = \int_{(e_y)} L^2(x_o, y) du(e_y) \text{ exists for all } y \text{ when } x_o \in (e_x).$$

$$(\beta) \quad L(y) \text{ and } |\phi^2(x)| \text{ are integrable } (B, u).$$

$$(\gamma) \quad \phi(x) \text{ is a solution of (3.1) with } f(y) = dF(e_y).$$

To prove this, equation (3.1) is multiplied by $du(e_y)$ and integrated over (e_y) . Interchange of limits is justified and comparison with (1.1) gives $d\psi(e_x) = \phi(x)$. Integrating this gives (3.2), as desired.

THEOREM. A solution of (3.1) exists and is given by

$$(3.3) \quad \phi(x) = d\psi(e_x)$$

provided that

$$(\alpha) \quad L^2(y) = \int_{(e_y)} L^2(x_o, y) du(e_y) \text{ exists for all } y \text{ when } x_o \in (e_x).$$

$$(\beta) \quad L(y) \text{ and } |d\psi(e_x)|^2 \text{ are integrable } (B, u).$$

$$(\gamma) \quad \psi(e_x) \text{ is a solution of (1.1) with } dF(e_y) = f(y).$$

The proof is obtained by substituting (3.3) and using (γ) in equation (1.1), and interchanging limits. Integrands of the resulting equation are the two members of (3.1).

Two solutions of (1.1) that differ by a singular function only, give rise to the same solution of (3.1). Conversely, a solution of (3.1) gives rise to a set of solutions of (1.1) which differ at most by a singular function.

IV. THE OPERATOR $S_x \{ \xi | h(x, e_y) \}$

$K(x, e_y)$ of equation (1.3) is measurable (B, u) , but it is not assumed that $K^2(x, e_y)$ is integrable (B, u) .

It is assumed that there exists a linear operator $S_x \{ \xi | h(x, e_y) \}$ having the parameter ξ , which meets the following conditions:

* (W.J.T.) Sections 12 and 13.

$$(4.1) \quad \int_{(D_x)} |S_x\{\xi|K(x, e_y)\}|^2 du(e_x) \text{ exists.}$$

$$(4.2) \quad |S_x\{\xi|K_n(x, e_y)\}| \leq \delta(\xi, y) \text{ where } \delta \text{ is independent of } n \text{ and } \delta^2 \text{ is integrable } (B, u) \text{ in } y.$$

$$(4.3) \quad \lim_{n \rightarrow \infty} S_x\{\xi|K_n(x, e_y)\} = S_x\{\xi|K(x, e_y)\}.$$

$$(4.4) \quad \lim_{n \rightarrow \infty} S_x\{\xi|F_n(e_y)\} = S_x\{\xi|F(e_y)\} \text{ whenever } \lim_{n \rightarrow \infty} F_n(e_y) = F(e_y).$$

$$(4.5) \quad \int_{(D_x)} S_x\{\xi|K_n(x, e_y)\} d\psi(e_x) = S_x\left\{\xi \left| \int_{(D_x)} K_n(x, e_y) d\psi(e_x) \right.\right\}.$$

Applying this operator to equation (1.5) gives

$$(4.6) \quad \int_{(D_x)} S_x\{\xi|K_n(x, e_y)\} \cdot d\psi_n(e_x) = S_x\{\xi|F_n(e_y)\}.$$

THEOREM. Two solutions of (4.6) can differ only by a singular function provided the equation

$$(4.7) \quad \int_{(D_x)} S_x\{\xi|K_n(x, e_y)\} d\psi_n(e_x) = 0$$

is closed, i.e., it has no non-singular solution different from zero.

The difference of any two solutions of (4.6) must satisfy (4.7) which can be true only if the two solutions differ by not more than a singular function.

A similar theorem holds for the equation obtained by applying the operator $S_x\{\xi|h(x, e_y)\}$ to equation (1.1).

V. SOLUTION OF EQUATIONS OF THE FIRST KIND

Applying operator $S_x\{\xi|h(x, e_y)\}$ to equation (1.1) gives

$$(5.1) \quad \int_{(D_x)} S_x\{\xi|K(x, e_y)\} d\psi(e_x) = S_x\{\xi|F(e_y)\}.$$

THEOREM. Let equation (4.7) be closed. Let there exist an infinite sequence of subscripts $\{n_j\} \subset \{n\}$ such that

$$\sum_{K=1}^{\infty} \lambda_{nk}^2 f_{nk}^2 \leq A < \infty$$

$$(5.2) \quad \sum_{K=1}^{\infty} \lambda_{nk}^{2+2\beta} f_{nk}^2 \leq B < \infty \text{ for some constant } \beta > 0$$

where B and A are independent of $\{n_j\}$ and

$$(5.3) \quad f_{nk} = \int_{(D_x)} f(z) \phi_{nk}(z) du(e_x) \text{ where } f(z) = dF(e_x).$$

Then there exists a solution of (5.1) given by

$$(5.4) \quad \psi(e_x) = \int_{-\infty}^{\infty} \lambda d_\lambda \left[\int_{(D_x)} f(z) d_x \theta(e_x, e_z | \lambda) du(e_z) \right]$$

where $\theta(e_x, e_z | \lambda)$ is the limit function in (2.2).

Proof. Let

$$(5.5) \quad \begin{aligned} \psi_n(e_x) &= \int_{-\infty}^{\infty} \lambda d_\lambda \left[\int_{(D_x)} f(z) d_x \theta_n(e_x, e_z | \lambda) du(e_z) \right] \\ &= \sum_{K=1}^{\infty} \lambda_{nk} f_{nk} \phi_{nk}(x) du(e_x) \end{aligned}$$

by (5.3) and (2.1). Bessel's and Schwartz's inequalities may be used to show that $|\psi_n(e_x)| \leq \sqrt{A} \cdot \sqrt{u(e_x)}$, and this bound is independent of n . The de la Vallee Poussin-Frostman theorem states that then there exists a subsequence $\{n_i\} \subset \{n\}$ and an additive function $\psi(e_x)$ such that on every Borel set on whose frontier $u(e_x)$ vanishes

$$\lim_{n_i \rightarrow \infty} \psi_{n_i}(e_x) = \psi(e_x) \text{ and } |\psi(e_x)| \leq \sqrt{A} \sqrt{u(e_x)}.$$

For such functions there exist* $\phi_n(x)$ and $\phi(x)$ such that almost everywhere one has

$$(5.6) \quad \phi_n(x) = d\psi_n(e_x) \text{ and } \phi(x) = d\psi(e_x).$$

By (5.6) and (5.5)

$$\int_{(e_x)} \phi_n(x) du(e_x) = \sum_{K=1}^{\infty} \lambda_{nk} f_{nk} \int_{(e_x)} \phi_{nk}(x) du(e_x).$$

By a theorem proved by Trjitzinsky,† then for some $\{n_i\} \subset \{n\}$

$$\phi_n(x) \sim \sum_{K=1}^{\infty} \lambda_{nk} f_{nk} \phi_{nk}(x).$$

It follows that the integral of $\phi_n(x)$ is bounded and $\phi_n(x)$ will converge weakly to $\phi(x)$. Also $\lim_{n_i \rightarrow \infty} \theta_{n_i}(x, y | \lambda) = \theta(x, y | \lambda)$.

Now with $\psi_n(e_x)$ as in (5.5), one uses (1.5) to define $F_n(e_y)$. One defines $F_n(l|e_y) = \int_{(D_x)} K_n(x, e_y) d\psi_n(l|e_x) = \int_{(e_x)} \sum_{K=1}^l \lambda_{nk} f_{nk} \phi_{nk}(x) du(e_x)$. Then $\psi_n(l|e_x) \sim \psi_n(e_x)$ as l approaches infinity.

$$(5.7) \quad F_n(e_y) = \lim_{l \rightarrow \infty} F(l|e_y) = \int_{(e_x)} \left[\sum_{K=1}^{\infty} f_{nk} \phi_{nk}(x) \right] du(e_x).$$

$$(5.8) \quad F_n(e_y) = \int_{-\infty}^{\infty} d_\lambda \left[\int_{(D_x)} f(z) d_x \theta_n(e_x, e_z | \lambda) du(e_z) \right] = \int_{-\infty}^{\infty} d_\lambda \sigma_n(e_x | \lambda).$$

* E. C. Titchmarsh, "Theory of Functions," Chap. 11.

† (W.J.T.) Section 13. Lemma 13.1.

If α is any constant greater than zero, then

$$F_n(e_y) = \left[\int_{-\infty}^{-\alpha} + \int_{-\alpha}^{\alpha} + \int_{\alpha}^{\infty} \right] d_{\lambda} \sigma_n(e_x | \lambda) = \int_{-\alpha}^{\alpha} d_{\lambda} \sigma_n(e_x | \lambda) + \frac{1}{\alpha} \sqrt{A} \cdot \sqrt{u(e_x)}.$$

Since A and $u(e_x)$ are independent of α then as $\alpha \rightarrow \infty$

$$F_n(e_y) = \int_{-\infty}^{\infty} d_{\lambda} \sigma_n(e_x | \lambda).$$

For some subsequence $\{n_j\}$, by means of (2.3) one has

$$(5.9) \quad \lim F_n(e_y) = \int_{-\infty}^{\infty} d_{\lambda} \sigma(e_x | \lambda) = \int_{(e_x)} f(x) du(e_x) = F(e_y)$$

almost everywhere by (5.3) and (5.7).

Now returning to equation (4.6), there exists a subsequence $\{n_j\}$ such that for n in this subsequence, by (4.4), (4.5) and (4.2)

$$\lim_{n \rightarrow \infty} S_x \{ \xi | F_n(e_y) \} = S_x \{ \xi | F(e_y) \} \quad \text{almost everywhere.}$$

$$\lim_{n \rightarrow \infty} S_x \left\{ \xi \left| \int_{(D_x)} K_n(x, e_y) \right. \right\} d\psi_n(e_x) = \int_{(D_x)} S_x \{ \xi | K(x, e_y) \} d\psi(e_x)$$

by (4.3) and (5.9). Therefore $d\psi(e_x)$, which equals $\phi(x)$ by (5.6) obtained from $\phi(x) = \lim \phi_n(x)$, satisfies equation (5.1) and $\psi(e_x) =$

$\int_{(e_x)} \phi(x) du(e_x)$ by (5.6). By (5.8), one has

$$\psi_n(e_x) = \left[\int_{-\infty}^{-\alpha} + \int_{-\alpha}^{\alpha} + \int_{\alpha}^{\infty} \right] \lambda d_{\lambda} \sigma_n(e_x) = \lim_{\alpha \rightarrow \infty} \left[\int_{-\alpha}^{\alpha} \lambda d_{\lambda} \sigma_n(e_x | \lambda) + \alpha^{(-\beta)} \sqrt{B} \sqrt{u(e_x)} \right]$$

where α and β are two positive constants. When n approaches ∞

$$(5.10) \quad \psi(e_x) = \int_{-\infty}^{\infty} \lambda d_{\lambda} \sigma(e_x | \lambda) = \int_{-\infty}^{\infty} \lambda d_{\lambda} \left[\int_{(D_x)} f(z) d_x \theta(e_x, e_y | \lambda) du(e_z) \right].$$

This is a solution of equation (5.1) for $\theta(e_x, e_y | \lambda)$ as in (2.1) and $f(z) = dF(e_z)$. The solution holds almost everywhere.

COROLLARY. If $K(x, e_y)$ is a kernel such that $K^2(x, e_y)$ is integrable (B, u) in (D_x) , then (5.10) is also a solution of (1.1).

The proof follows from the fact that the identity is then one of the operators $S_x \{ \xi | h(x, e_y) \}$. For the identity operator equations (5.1) and (1.1) are the same.

VITA

Corinne Rose Hattan was born November 2, 1903, in Mt. Hope, Kansas. She was graduated from Mt. Hope High School in 1921. She attended Ottawa University, receiving the B.S. degree in 1925. After two years of teaching in high school, she became a graduate assistant in mathematics at the University of Kansas, receiving the M.A. degree in 1928. She was an Instructor in Mathematics at the University of Kansas until 1931, and also from 1935 to 1937. For one year she filled a temporary position as Assistant Professor of Mathematics at Southwestern College. From 1937 to 1941 she was Head of the Mathematics and Biology Departments of Northwestern Junior College, Orange City, Iowa. She was a Graduate Assistant in Mathematics at the University of Illinois from 1941 to 1945.

She is a member of Sigma Xi, Phi Beta Kappa, Pi Mu Epsilon, the Mathematical Association of America, and the American Mathematical Society.

